Using Alternative Strain Measures in a User Material

```
subroutine umat(stress,sdv,ddsdde,sse,spd,scd,rpl,ddsddt,                  &
                drplde,drpldt,stran,dstran,time,dtime,temp,dtemp,predef,dpred, &
                cmname,ndi,nshr,ntens,numsdv,props,nprop, coords,drot,pnewdt,  &
                celent,dfgrd0,dfgrd1,noel,npt,layer,kspt,kstep,kinc)
    ! ..
    use kPrecision    , only : rp
    use kConstants    , only : zero, one, two
    ! ..
    implicit none
    ! ..
    ! Parameters
    integer, parameter :: iStepTime        = 1,    &
                          iTotalTime       = 2,    &
                          nTime            = 2
    ! ..
    ! Arguments
    ! ..
    character*80,intent(in)      :: cmname
    real(rp), intent(inout)      :: stress(ntens), sdv(numsdv)
    real(rp), intent(inout)      :: ddsddt(ntens), ddsdde(ntens, ntens)
    real(rp), intent(inout)      :: drplde(ntens), drpldt
    real(rp), intent(inout)      :: sse, spd, scd, rpl, pnewdt
    real(rp), intent(in)         :: stran(ntens), dstran(ntens)
    real(rp), intent(in)         :: time(2)     , dtime
    real(rp), intent(in)         :: temp        , dtemp
    real(rp), intent(in)         :: predef(2)   , dpred(2)
    real(rp), target, intent(in) :: props(nprop)
    real(rp), intent(in)         ::  coords(3)  , drot(3,3)
    real(rp), intent(in)         :: dfgrd0(3,3) , dfgrd1(3,3), celent
    integer,  intent(in)         :: ndi, nshr, ntens, numsdv, nprop
    integer,  intent(in)         :: noel,npt,layer,kspt,kstep,kinc
    ! ..
    ! Locals
    ! ..
    real(rp) :: E, nu, G, K
    integer  :: i
    ! ..
    ! .. Code
    E  = 210D3
    nu = 0.3D0
    G  = E/(two*(one+nu))
    K  = E/(one-two*nu)
    ! ..
    ddsdde = zero
    ddsdde(1:ndi, 1:ndi) = K
    do i = 1, ndi
        ddsdde(i,i) = ddsdde(i,i) + two*G
    end do
    do i = ndi+1, ndi+nshr
        ddsdde(i,i) = G
    end do
```

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Executive Summary

When implementing (anisotropic) hyperelastic materials (or materials with strain measures alternative to the logarithmic strain) in a user subroutine, it may be necessary to use a different strain in the derivation. However, we must still provide Abaqus with the proper derivative it requires; specifically, the material Jacobian ***matrix*** $\partial\Delta\sigma/\partial\Delta\varepsilon$. In this document we show how such a derivative can be obtained.

The intended audience for this document is primarily those users of Abaqus/Standard that develop UMAT subroutines; however, all users of Abaqus with an interest in advanced mechanics will benefit.

Contents

1. Introduction

In Abaqus, user subroutine UMAT requires one to compute the derivative of the Cauchy stress $\boldsymbol{\sigma}$ with respect to the strain increment $\Delta\boldsymbol{\varepsilon}$. The strain increment in Abaqus is defined as the increment in the rate of deformation tensor, which will be denoted by $\Delta\boldsymbol{D}$, which naturally leads, when integrated in non-shearing problems, to a logarithmic strain.

If we want to use any other strain measure, or even use an arbitrary material model which depends solely on the deformation gradient, it would be helpful to have a generic way of finding the correct derivative of stress, even if we have defined the stress using another strain measure. Everything will then become a matter of using the chain rule in the correct way.

In this paper we will present a methodology to use any strain measure, provided it is computed from the deformation gradient $\boldsymbol{F}$. In Section 2 we will introduce the notation and in Section 3 we will introduce additional notation for constructing fourth order tensors. This will help us in the subsequent derivation of derivatives of tensors. We then have a brief discussion about subspaces in Section 4. The derivative of the velocity gradient with respect to the deformation gradient is obtained in Section 5. This then finally culminates in being able to obtain the derivative of the deformation gradient with respect to the rate of deformation tensor in Section 6. And since any deformation measure used in an Abaqus user-defined material is implicitly derived from the deformation gradient, this will give us the possibility to use any arbitrary strain measure in a user material.

The development of a UMAT user subroutine generally requires considerable expertise. The implementation of any realistic constitutive model requires extensive development and testing. It is highly recommended that you familiarize yourself with the relevant portions of the Abaqus documentation before undertaking the development of a user material model. A list of references is supplied at the end of this document.

2. Notation

All tensors in this paper are defined with respect to a fixed orthonormal basis. This simplifies the discussion by eliminating the need to take derivatives of the basis vectors. The basis vectors are denoted by $\boldsymbol{e}_i$, where typically $i \in \{1,2,3\}$. Tensors are denoted by order, where the order denotes the number of independent sets of basis vectors. Thus a vector, which is a first order tensor, has only a single independent set of basis vectors:

$$\boldsymbol{v} = \sum_{i=1}^{3} v_i \boldsymbol{e}_{\boldsymbol{i}},$$

whereas a second order tensor has two sets:

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} \boldsymbol{e}_i \boldsymbol{e}_j.$$

Furthermore, we are going to use Einstein notation, which means we drop the summation symbols and it is assumed a summation occurs if we have a repeated index.

$$\boldsymbol{\sigma} = \sigma_{ij} \boldsymbol{e}_i \boldsymbol{e}_j$$

And we will drop the basis vectors, if the order of it is clear:

$$\boldsymbol{\sigma} = \sigma_{ij}$$

where we assume that the first index is associated with the first set of basis vectors and the second index is associated with the second set of basis vectors. The alphabetical order of the indices is not relevant; thus we may write

$$\boldsymbol{\sigma}_{ji} = \sigma_{ji} \boldsymbol{e}_j \boldsymbol{e}_i$$

To write the transpose, we need to include the basis vectors

$$\sigma_{ji} \boldsymbol{e}_i \boldsymbol{e}_j,$$

or introduce another tensor which represents the transpose: Define

$$\boldsymbol{\sigma}^T = \sigma_{ji} \boldsymbol{e}_i \boldsymbol{e}_j.$$

We will introduce notation that facilitates obtaining derivatives for second order tensors. In particular, we are going to address the derivatives that are required in most finite element solvers.

In Sections 3 and 4, vectors will be written by bold lower case letters: for example $\boldsymbol{v}$ and $\boldsymbol{w}$; second order tensors by bold upper case letters: for example $\boldsymbol{A}$ and $\boldsymbol{B}$; and fourth order tensors by bold calligraphic uppercase letters: for example $\boldsymbol{\mathcal{I}}$ and $\boldsymbol{\mathcal{D}}$.

3. Fourth Order Tensor Constructors

The dyadic product operator is a standard method for creating higher order tensors from lower order tensors.

Definition 1. Let $\boldsymbol{v} = v_i \boldsymbol{e}_i$ and $\boldsymbol{w} = w_i \boldsymbol{e}_i$ be two first order tensors (vectors). Then their dyadic product $\boldsymbol{A} = \boldsymbol{v} \otimes \boldsymbol{w}$ is defined as:

$$A_{ij} \boldsymbol{e}_i \boldsymbol{e}_j = v_i w_j \boldsymbol{e}_i \boldsymbol{e}_j$$

The dyadic product is often written without explicitly referring to the outer product operator $\otimes$, thus $\boldsymbol{v} \otimes \boldsymbol{w} = \boldsymbol{v}\boldsymbol{w}$. We can generalize this to higher dimensions. Note that the dyadic product does not change the order of the base vectors. Thus,

$$\boldsymbol{A} \otimes \boldsymbol{B} = A_{ij} B_{kl} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k \boldsymbol{e}_l$$

We introduce two additional notations $\boldsymbol{A} \looparrowright \boldsymbol{B}$ and $\curvearrowright \boldsymbol{B}$. The reason the operators are chosen in this way is to show the way the higher order tensor is constructed from the lower order ones.

Definition 2. (Weave Constructor) The weave constructor takes two second order tensors and constructs a fourth order tensor as:

$$\boldsymbol{A} \looparrowright \boldsymbol{B} = A_{ik} B_{jl} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k \boldsymbol{e}_l$$

Definition 3. (Clamp Constructor) The clamp constructor takes two second order tensors and constructs a fourth order tensor as:

$$\boldsymbol{A} \curvearrowright \boldsymbol{B} = A_{il} B_{jk} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k \boldsymbol{e}_l$$

These operations are essentially generalized dyadic products that work on second order tensors.

Let us specifically take $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$ and $\boldsymbol{B} = \boldsymbol{c} \otimes \boldsymbol{d}$. These are second order tensors, constructed from the regular dyadic product with vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$.

Then the weave operation is a generalized dyadic product that changes the order of the vectors in the dyad in the following way:

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \looparrowright (\boldsymbol{c} \otimes \boldsymbol{d}) = \boldsymbol{a} \otimes \boldsymbol{c} \otimes \boldsymbol{b} \otimes \boldsymbol{d}$$

This also explains the name we choose for the operator, since it weaves the first second order tensor into the second one by putting the second "leg" of tensor $\boldsymbol{A}$, (vector $\boldsymbol{b}$) inside the two legs of tensor $\boldsymbol{B}$ (vectors $\boldsymbol{c}$ and $\boldsymbol{d}$).

In a similar way we can look at the clamp operator, which operates on the same second order tensors in the following way:

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \curvearrowright (\boldsymbol{c} \otimes \boldsymbol{d}) = \boldsymbol{a} \otimes \boldsymbol{c} \otimes \boldsymbol{d} \otimes \boldsymbol{b}$$

So in this case the second "leg" of tensor $\boldsymbol{A}$, which is the (vector $\boldsymbol{b}$) is moved completely over tensor $\boldsymbol{B}$, which is now sitting in the middle. Hence, $\boldsymbol{B}$ is clamped by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

The above definitions generalize this operation for sums of dyads.

Proposition 1. Fourth order tensors constructed using the weave and clamp constructors have the following interactions with each other when we apply a double contraction between them:

$$
\begin{aligned}
(\boldsymbol{A} \looparrowright \boldsymbol{B}):(\boldsymbol{C} \looparrowright \boldsymbol{D}) &= (\boldsymbol{A} \cdot \boldsymbol{C}) \looparrowright (\boldsymbol{B} \cdot \boldsymbol{D}) \\
(\boldsymbol{A} \looparrowright \boldsymbol{B}):(\boldsymbol{C} \curvearrowright \boldsymbol{D}) &= (\boldsymbol{A} \cdot \boldsymbol{C}) \curvearrowright (\boldsymbol{B} \cdot \boldsymbol{D}) \\
(\boldsymbol{A} \curvearrowright \boldsymbol{B}):(\boldsymbol{C} \looparrowright \boldsymbol{D}) &= (\boldsymbol{A} \cdot \boldsymbol{D}) \curvearrowright (\boldsymbol{B} \cdot \boldsymbol{C}) \\
(\boldsymbol{A} \curvearrowright \boldsymbol{B}):(\boldsymbol{C} \curvearrowright \boldsymbol{D}) &= (\boldsymbol{A} \cdot \boldsymbol{D}) \looparrowright (\boldsymbol{B} \cdot \boldsymbol{C})
\end{aligned}
$$

Proof: We start with

$$
\begin{aligned}
&(\boldsymbol{A} \looparrowright \boldsymbol{B}):(\boldsymbol{C} \looparrowright \boldsymbol{D}) = A_{ik}B_{jl}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l : C_{pr}D_{qs}\boldsymbol{e}_p\boldsymbol{e}_q\boldsymbol{e}_r\boldsymbol{e}_s \\
&= A_{ik}B_{jl}\delta_{kp}\delta_{lq}C_{pr}D_{qs}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_r\boldsymbol{e}_s \\
&= A_{ip}B_{jq}C_{pr}D_{qs}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_r\boldsymbol{e}_s \\
&= (A_{ip}C_{pr})(B_{jq}D_{qs})\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_r\boldsymbol{e}_s \\
&= (\boldsymbol{A} \cdot \boldsymbol{C}) \looparrowright (\boldsymbol{B} \cdot \boldsymbol{D})
\end{aligned}
$$

In the same way we can prove the second equation

$$
\begin{aligned}
&(\boldsymbol{A} \looparrowright \boldsymbol{B}):(\boldsymbol{C} \curvearrowright \boldsymbol{D}) = A_{ik}B_{jl}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l : C_{ps}D_{qr}\boldsymbol{e}_p\boldsymbol{e}_q\boldsymbol{e}_r\boldsymbol{e}_s \\
&= A_{ik}B_{jl}\delta_{kp}\delta_{lq}C_{pr}D_{qs}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_r\boldsymbol{e}_s \\
&= A_{ip}B_{jq}C_{ps}D_{qr}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_r\boldsymbol{e}_s \\
&= (A_{ip}C_{ps})(B_{jq}D_{qr})\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_r\boldsymbol{e}_s \\
&= (\boldsymbol{A} \cdot \boldsymbol{C}) \curvearrowright (\boldsymbol{B} \cdot \boldsymbol{D})
\end{aligned}
$$

And then the third equation

$$
\begin{aligned}
&(\boldsymbol{A} \curvearrowright \boldsymbol{B}):(\boldsymbol{C} \looparrowright \boldsymbol{D}) = A_{il}B_{jk}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l : C_{pr}D_{qs}\boldsymbol{e}_p\boldsymbol{e}_q\boldsymbol{e}_r\boldsymbol{e}_s \\
&= A_{il}B_{jk}\delta_{kp}\delta_{lq}C_{pr}D_{qs}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_r\boldsymbol{e}_s \\
&= A_{iq}B_{jp}C_{pr}D_{qs}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_r\boldsymbol{e}_s \\
&= (A_{ip}D_{qs})(B_{jp}C_{pr})\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_r\boldsymbol{e}_s \\
&= (\boldsymbol{A} \cdot \boldsymbol{D}) \curvearrowright (\boldsymbol{B} \cdot \boldsymbol{C})
\end{aligned}
$$

And finally the last equation

$$
\begin{aligned}
&(\boldsymbol{A} \curvearrowright \boldsymbol{B}):(\boldsymbol{C} \curvearrowright \boldsymbol{D}) = A_{il}B_{jk}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l : C_{ps}D_{qr}\boldsymbol{e}_p\boldsymbol{e}_q\boldsymbol{e}_r\boldsymbol{e}_s \\
&= A_{il}B_{jk}\delta_{kp}\delta_{lq}C_{ps}D_{qr}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_r\boldsymbol{e}_s \\
&= A_{iq}B_{jp}C_{ps}D_{qr}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_r\boldsymbol{e}_s \\
&= (A_{iq}D_{qr})(B_{jp}C_{ps})\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_r\boldsymbol{e}_s \\
&= (\boldsymbol{A} \cdot \boldsymbol{D}) \looparrowright (\boldsymbol{B} \cdot \boldsymbol{C})
\end{aligned}
$$

Which completes the proof.

From this we can see that the operators behave analogously to signed quantities: weave times weave is weave, clamp times clamp is weave, and mixed weave and clamp become clamp. These are useful to remember, since they make working with the operators quite convenient.

The next Proposition deals with the newly introduced operators working with second order tensors.

Proposition 2. Fourth order tensors constructed using the weave and clamp constructors have the following interactions when applying a double contraction with a second order tensor:

$$
\begin{aligned}
(\boldsymbol{A} \looparrowright \boldsymbol{B}) : \boldsymbol{C} &= \boldsymbol{A} \cdot \boldsymbol{C} \cdot \boldsymbol{B}^T \\
(\boldsymbol{A} \curvearrowright \boldsymbol{B}) : \boldsymbol{C} &= \boldsymbol{A} \cdot \boldsymbol{C}^T \cdot \boldsymbol{B}^T \\
\boldsymbol{C} : (\boldsymbol{A} \looparrowright \boldsymbol{B}) &= \boldsymbol{A}^T \cdot \boldsymbol{C} \cdot \boldsymbol{B} \\
\boldsymbol{C} : (\boldsymbol{A} \curvearrowright \boldsymbol{B}) &= \boldsymbol{B}^T \cdot \boldsymbol{C}^T \cdot \boldsymbol{A}
\end{aligned}
$$

Proof: Let us again start with the first equation, and write it out fully

$$
\begin{aligned}
(\boldsymbol{A} \looparrowright \boldsymbol{B}) : \boldsymbol{C} &= A_{ik} B_{jl} C_{kl} \boldsymbol{e}_i \boldsymbol{e}_j \\
&= \boldsymbol{A} \cdot \boldsymbol{C} \cdot \boldsymbol{B}^T
\end{aligned}
$$

Onto the next, which follows a similar fashion

$$
\begin{aligned}
(\boldsymbol{A} \curvearrowright \boldsymbol{B}) : \boldsymbol{C} &= A_{il} B_{jk} C_{kl} \boldsymbol{e}_i \boldsymbol{e}_j \\
&= \boldsymbol{A} \cdot \boldsymbol{C}^T \cdot \boldsymbol{B}^T
\end{aligned}
$$

Then number three

$$
\begin{aligned}
\boldsymbol{C} : (\boldsymbol{A} \looparrowright \boldsymbol{B}) &= C_{ij} A_{ik} B_{jl} \boldsymbol{e}_k \boldsymbol{e}_l \\
&= \boldsymbol{A}^T \cdot \boldsymbol{C} \cdot \boldsymbol{B}
\end{aligned}
$$

And finally the last one

$$
\begin{aligned}
\boldsymbol{C} : (\boldsymbol{A} \curvearrowright \boldsymbol{B}) &= C_{ij} A_{il} B_{jk} \boldsymbol{e}_k \boldsymbol{e}_l \\
&= \boldsymbol{B}^T \cdot \boldsymbol{C}^T \cdot \boldsymbol{A}
\end{aligned}
$$

Which concludes the proof.

Again these definitions are quite useful, since it allows us to “bring” a second order tensor to the end.

4. Subspaces of Second Order Tensors

Tensors form a vector space, and as such are given with respect to a basis. The basis for symmetric tensors has six basis 'tensors', whereas the basis for general tensors has nine basis tensors. For now, we are not going to introduce any symmetry argument, but these are going to be of importance later if we need to invert the derivatives of some tensors.

Definition 4. $\boldsymbol{I}$ is the second order unit tensor given by

$$\boldsymbol{I} = \delta_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$$

In a similar way we define the fourth order unit tensor.

Definition 5. $\boldsymbol{\mathcal{I}}$ is the fourth order unit tensor given by

$$\begin{aligned}\boldsymbol{\mathcal{I}} &= \delta_{ik}\delta_{jl}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l \\ &= \boldsymbol{I} \looparrowright \boldsymbol{I}\end{aligned}$$

Theorem 1. The derivative of a second order tensor with respect to itself is $\boldsymbol{I} \looparrowright \boldsymbol{I}$.

Proof: Let $\boldsymbol{A}$ be an arbitrary second order tensor with nine independent components. Then

$$\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{A}} = \frac{\partial A_{ij}}{\partial A_{kl}}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l$$

Now since all the A_{ij} are independent components, this is only non-zero if $i = k$ and $j = l$, so

$$\frac{\partial A_{ij}}{\partial A_{kl}} = \delta_{ik}\delta_{jl}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l.$$

Definition 6. Let $\boldsymbol{A}$ be an arbitrary second order tensor. The symmetric part of $\boldsymbol{A}$ is:

$$\mathrm{sym}(\boldsymbol{A}) = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T).$$

Proposition 3. Let $\boldsymbol{A}$ be an arbitrary second order tensor. The symmetric part of $\boldsymbol{A}$ can be determined by

$$\mathrm{sym}(\boldsymbol{A}) = \frac{1}{2}\big((\boldsymbol{I} \looparrowright \boldsymbol{I}) + (\boldsymbol{I} \curvearrowright \boldsymbol{I})\big) : \boldsymbol{A}$$

Proof: We have

$$\begin{aligned}\mathrm{sym}(\mathbf{A}) &= \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T) \\ &= \frac{1}{2}(\boldsymbol{I} \cdot \boldsymbol{A} \cdot \boldsymbol{I} \ + \boldsymbol{I} \cdot \boldsymbol{A}^T \cdot \boldsymbol{I}) \\ &= \frac{1}{2}\big((\boldsymbol{I} \looparrowright \boldsymbol{I}) : \boldsymbol{A} + (\boldsymbol{I} \curvearrowright \boldsymbol{I}) : \boldsymbol{A}\big) \\ &= \frac{1}{2}\big((\boldsymbol{I} \looparrowright \boldsymbol{I}) + (\boldsymbol{I} \curvearrowright \boldsymbol{I})\big) : \boldsymbol{A}\end{aligned}$$

Where we used parts 1 and 2 of Proposition 2.

Definition 7. Let $\boldsymbol{A}$ be an arbitrary second order tensor. The asymmetric part of $\boldsymbol{A}$ is:

$$\operatorname{asym}(\boldsymbol{A}) = \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T)$$

Proposition 4. Let $\boldsymbol{A}$ be an arbitrary second order tensor. The asymmetric part of $\boldsymbol{A}$ can be determined by

$$\operatorname{asym}(\boldsymbol{A}) = \frac{1}{2}\big((\boldsymbol{I} \looparrowright \boldsymbol{I}) - (\boldsymbol{I} \curvearrowright \boldsymbol{I})\big): \boldsymbol{A}$$

Proof: We have

$$\begin{aligned}
\operatorname{asym}(\boldsymbol{A}) &= \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T) \\
&= \frac{1}{2}(\boldsymbol{I} \cdot \boldsymbol{A} \cdot \boldsymbol{I} - \boldsymbol{I} \cdot \boldsymbol{A}^T \cdot \boldsymbol{I}) \\
&= \frac{1}{2}\big((\boldsymbol{I} \looparrowright \boldsymbol{I}): \boldsymbol{A} - (\boldsymbol{I} \curvearrowright \boldsymbol{I}): \boldsymbol{A}\big) \\
&= \frac{1}{2}\big((\boldsymbol{I} \looparrowright \boldsymbol{I}) - (\boldsymbol{I} \curvearrowright \boldsymbol{I})\big): \boldsymbol{A}
\end{aligned}$$

Where we again used parts 1 and 2 of Proposition 2.

We are going to assume operator precedence of the weave and clamp over addition and subtraction, so we can write the above results as

$$\begin{aligned}
\operatorname{sym}(\boldsymbol{A}) &= \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} + \boldsymbol{I} \curvearrowright \boldsymbol{I}): \boldsymbol{A} \\
\operatorname{asym}(\boldsymbol{A}) &= \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} - \boldsymbol{I} \curvearrowright \boldsymbol{I}): \boldsymbol{A}
\end{aligned}$$

It is clear that the symmetric part of a symmetric tensor is the tensor itself, thus if $\boldsymbol{A} = \boldsymbol{A}^T$ then $\operatorname{sym}(\boldsymbol{A}) = \boldsymbol{A}$ and $\operatorname{asym}(\boldsymbol{A}) = \boldsymbol{0}$.

Corollary 1. The derivative of the symmetric operator is given by

$$\frac{\partial \operatorname{sym}(\boldsymbol{A})}{\partial \boldsymbol{A}} = \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} + \boldsymbol{I} \curvearrowright \boldsymbol{I})$$

Proof: Let $\boldsymbol{A}$ be any tensor. Then we have

$$\begin{aligned}
\frac{\partial \operatorname{sym}(\boldsymbol{A})}{\partial \boldsymbol{A}} &= \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} + \boldsymbol{I} \curvearrowright \boldsymbol{I}): \frac{\partial \boldsymbol{A}}{\partial \boldsymbol{A}} \\
&= \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} + \boldsymbol{I} \curvearrowright \boldsymbol{I}): (\boldsymbol{I} \looparrowright \boldsymbol{I}) \\
&= \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} + \boldsymbol{I} \curvearrowright \boldsymbol{I})
\end{aligned}$$

Where we used parts 1 and 2 of Proposition 1.

Corollary 2. The derivative of the asymmetric operator is given by

$$\frac{\partial \mathrm{asym}(\boldsymbol{A})}{\partial \boldsymbol{A}} = \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} - \boldsymbol{I} \curvearrowright \boldsymbol{I})$$

Proof: Let $\boldsymbol{A}$ be any tensor. Then we have

$$\begin{aligned}\frac{\partial \mathrm{asym}(\boldsymbol{A})}{\partial \boldsymbol{A}} &= \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} - \boldsymbol{I} \curvearrowright \boldsymbol{I}) : \frac{\partial \boldsymbol{A}}{\partial \boldsymbol{A}} \\ &= \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} - \boldsymbol{I} \curvearrowright \boldsymbol{I}) : (\boldsymbol{I} \looparrowright \boldsymbol{I}) \\ &= \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} - \boldsymbol{I} \curvearrowright \boldsymbol{I})\end{aligned}$$

Where we used parts 1 and 2 of Proposition 1.

Proposition 5. Let A be an arbitrary second order tensor. Then $\mathrm{sym}\big(\mathrm{asym}(\boldsymbol{A})\big) = \mathrm{asym}\big(\mathrm{sym}(\mathrm{A})\big) = \boldsymbol{0}$, where $\boldsymbol{0}$ is the second order null tensor, thus all its components are zero.

Proof: This follows straight from the definition of the operators and application of Proposition 1:

$$\begin{aligned}\mathrm{sym}\big(\mathrm{asym}(\boldsymbol{A})\big) &= \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} + \boldsymbol{I} \curvearrowright \boldsymbol{I}) : \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} - \boldsymbol{I} \curvearrowright \boldsymbol{I}) : \boldsymbol{A} \\ &= \frac{1}{4}\big((\boldsymbol{I} \looparrowright \boldsymbol{I}) : (\boldsymbol{I} \looparrowright \boldsymbol{I} - \boldsymbol{I} \curvearrowright \boldsymbol{I}) + (\boldsymbol{I} \curvearrowright \boldsymbol{I}) : (\boldsymbol{I} \looparrowright \boldsymbol{I} - \boldsymbol{I} \curvearrowright \boldsymbol{I})\big) : \boldsymbol{A} \\ &= \frac{1}{4}\big((\boldsymbol{I} \looparrowright \boldsymbol{I} - \boldsymbol{I} \curvearrowright \boldsymbol{I}) + (\boldsymbol{I} \curvearrowright \boldsymbol{I} - \boldsymbol{I} \looparrowright \boldsymbol{I})\big) : \boldsymbol{A} \\ &= \frac{1}{4}\boldsymbol{\mathcal{O}} : \boldsymbol{A} \\ &= \boldsymbol{0}\end{aligned}$$

The other proofs run in exactly the same vein, we copy the definitions, and then apply the rules of Proposition 1.

Proposition 6. Let A be an arbitrary second order tensor. Then $\mathrm{sym}\big(\mathrm{sym}(\boldsymbol{A})\big) = \mathrm{sym}(\boldsymbol{A})$

Proof: This follows straight from the definition of the operators and application of the rules of Proposition 1.

Proposition 7. Let A be an arbitrary second order tensor. Then $\mathrm{asym}\big(\mathrm{asym}(\boldsymbol{A})\big) = \mathrm{asym}(\boldsymbol{A})$

Proof: This follows straight from the definition of the operators and application of the rules of Proposition 1.

Finally, it is often said that the derivative of a symmetric tensor is given by $\frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} + \boldsymbol{I} \curvearrowright \boldsymbol{I})$, but we wish to stress that this is technically incorrect. In fact, this is the derivative of the symmetry function, as it operates between the full space of second order tensors and itself. If we consider symmetric tensors, then this is a proper subspace of the full space of second order tensors, and we should also look at the restriction of the derivative operator. However, we would need to change bases and our fourth order tensors lose their familiar form. Suffice it to say, that from the perspective of the full space $\alpha(\boldsymbol{I} \looparrowright \boldsymbol{I}) + (1-\alpha)(\boldsymbol{I} \curvearrowright \boldsymbol{I})$ are all proper derivatives of a symmetric tensor with respect to itself, and thus we lose unicity. From the viewpoint of the restricted symmetric space, all these operators are essentially the same: no matter what basis we choose for the symmetric subspace, the components will always be the same, regardless of the choice of α.

5. The Derivative of ΔL

Definition 8. Let $\dot{\boldsymbol{F}} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{t}}$. The velocity gradient is then given by

$$\boldsymbol{L} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}.$$

The velocity gradient will be used to define the rate of deformation tensor and the spin tensor. The integral of the rate of deformation tensor over a specific time increment is used by Abaqus as the strain increment. To be precise

$$\begin{aligned} \boldsymbol{D} &= \mathrm{sym}(\boldsymbol{L}) \\ \boldsymbol{W} &= \mathrm{asym}(\boldsymbol{L}) \end{aligned}$$

For ease of notation we now introduce the following time integrals, which are per increment, where t_0 is the beginning of the increment and t_1 is the end of the increment.

$$\begin{aligned} \Delta\boldsymbol{L} &= \int_{t_0}^{t_1} \boldsymbol{L} dt \\ \Delta\boldsymbol{D} &= \int_{t_0}^{t_1} \boldsymbol{D} dt = \mathrm{sym}(\Delta\boldsymbol{L}) \\ \Delta\boldsymbol{W} &= \int_{t_0}^{t_1} \boldsymbol{W} dt = \mathrm{asym}(\Delta\boldsymbol{L}) \end{aligned}$$

Abaqus uses $\Delta\boldsymbol{\varepsilon} = \Delta\boldsymbol{D}$ as the strain increment passed into UMAT.

We thus need to specify the way in which the above time integral of $\boldsymbol{L}$ is going to be computed. For this we need additional notation: the deformation gradient at the beginning of the increment is written as $\boldsymbol{F}_0$ and at the end of the increment as $\boldsymbol{F}$.

The increment in the velocity gradient in Abaqus is approximated by

$$\Delta\boldsymbol{L} = \Delta\boldsymbol{F} \cdot \widehat{\boldsymbol{F}}^{-1},$$

where $\widehat{\boldsymbol{F}}$ is the average deformation gradient over the increment, and $\Delta\boldsymbol{F}$ is the increment of the deformation gradient. Both are given in terms of the deformation gradient at the beginning and end of the increment as:

$$\begin{aligned}\widehat{\boldsymbol{F}} &= \frac{1}{2}(\boldsymbol{F}_0 + \boldsymbol{F})\\ \Delta\boldsymbol{F} &= \boldsymbol{F} - \boldsymbol{F}_0\end{aligned}$$

Before we continue our discussion, we present a number of propositions on the derivative of tensors.

Proposition 8. The derivative of a product of tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ with respect to another tensor $\boldsymbol{X}$ is given by

$$\frac{\partial \boldsymbol{A}\cdot\boldsymbol{B}}{\partial\boldsymbol{X}} = (\boldsymbol{I} \looparrowright \boldsymbol{B}^T){:}\frac{\partial\boldsymbol{A}}{\partial\boldsymbol{X}} + (\boldsymbol{A} \looparrowright \boldsymbol{I}){:}\frac{\partial\boldsymbol{B}}{\partial\boldsymbol{X}}.$$

Proof: We begin by using variations

$$\begin{aligned}\delta(\boldsymbol{A}\cdot\boldsymbol{B}) &= \delta\boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}\cdot\delta\boldsymbol{B}\\ &= \boldsymbol{I}\cdot\delta\boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}\cdot\delta\boldsymbol{B}\\ &= (\boldsymbol{I} \looparrowright \boldsymbol{B}^T){:}\,\delta\boldsymbol{A} + \boldsymbol{A}\cdot(\boldsymbol{I} \looparrowright \boldsymbol{I}){:}\,\delta\boldsymbol{B}\\ &= (\boldsymbol{I} \looparrowright \boldsymbol{B}^T){:}\,\delta\boldsymbol{A} + (\boldsymbol{A} \looparrowright \boldsymbol{I}){:}\,\delta\boldsymbol{B}\end{aligned}$$

From this we find

$$\frac{\partial \boldsymbol{A}\cdot\boldsymbol{B}}{\partial\boldsymbol{X}} = (\boldsymbol{I} \looparrowright \boldsymbol{B}^T){:}\frac{\partial\boldsymbol{A}}{\partial\boldsymbol{X}} + (\boldsymbol{A} \looparrowright \boldsymbol{I}){:}\frac{\partial\boldsymbol{B}}{\partial\boldsymbol{X}}$$

which concludes the proof.

Proposition 9. The derivative of the inverse of an invertible tensor A is given by

$$\frac{\partial\boldsymbol{A}^{-1}}{\partial\boldsymbol{X}} = -(\boldsymbol{A}^{-1} \looparrowright \boldsymbol{A}^{-T}){:}\frac{\partial\boldsymbol{A}}{\partial\boldsymbol{X}}.$$

Proof: Start from the identity $\boldsymbol{A}^{-1}\cdot\boldsymbol{A} = \boldsymbol{I}$, and take a variation:

$$\begin{aligned}\delta(\boldsymbol{A}^{-1})\cdot\boldsymbol{A} + \boldsymbol{A}^{-1}\cdot\delta\boldsymbol{A} &= \boldsymbol{0} && \Leftrightarrow\\ \delta(\boldsymbol{A}^{-1})\cdot\boldsymbol{A} &= -\boldsymbol{A}^{-1}\cdot\delta\boldsymbol{A} && \Leftrightarrow\\ \delta(\boldsymbol{A}^{-1}) &= -\boldsymbol{A}^{-1}\cdot\delta\boldsymbol{A}\cdot\boldsymbol{A}^{-1} && \Leftrightarrow\\ \delta(\boldsymbol{A}^{-1}) &= -(\boldsymbol{A}^{-1} \looparrowright \boldsymbol{A}^{-T}){:}\,\delta\boldsymbol{A}\end{aligned}$$

Since the variations are arbitrary on the left and right we get the required result.

Corollary 3. The derivatives of $\widehat{\boldsymbol{F}}$ and $\boldsymbol{\Delta F}$ with respect to $\boldsymbol{F}$ are given by

$$\frac{\partial \widehat{\boldsymbol{F}}}{\partial \boldsymbol{F}} = \frac{1}{2}\boldsymbol{I} \looparrowright \mathbf{I}$$
$$\frac{\partial \Delta \boldsymbol{F}}{\partial \boldsymbol{F}} = \boldsymbol{I} \looparrowright \boldsymbol{I}$$

Corollary 4. The derivative of $\widehat{\boldsymbol{F}}^{-1}$ with respect to $\boldsymbol{F}$ is given as:

$$\frac{\partial \widehat{\boldsymbol{F}}^{-1}}{\partial \boldsymbol{F}} = -\frac{1}{2}\left(\widehat{\boldsymbol{F}}^{-1} \looparrowright \widehat{\boldsymbol{F}}^{-T}\right).$$

Proof: We can combine the previous results to get

$$\begin{aligned}\frac{\partial \widehat{\boldsymbol{F}}^{-1}}{\partial \boldsymbol{F}} &= -\left(\widehat{\boldsymbol{F}}^{-1} \looparrowright \widehat{\boldsymbol{F}}^{-T}\right):\frac{\partial \widehat{\boldsymbol{F}}}{\partial \boldsymbol{F}} \\ &= -\left(\widehat{\boldsymbol{F}}^{-1} \looparrowright \widehat{\boldsymbol{F}}^{-T}\right):\left(\frac{1}{2}\boldsymbol{I} \looparrowright \mathbf{I}\right) \\ &= -\frac{1}{2}\left(\widehat{\boldsymbol{F}}^{-1} \looparrowright \widehat{\boldsymbol{F}}^{-T}\right).\end{aligned}$$

Proposition 10. The derivative of the increment in the velocity gradient with respect to $\boldsymbol{F}$ is given by

$$\frac{\partial \Delta \boldsymbol{L}}{\partial \boldsymbol{F}} = \boldsymbol{F}_0 \cdot \widehat{\boldsymbol{F}}^{-1} \looparrowright \widehat{\boldsymbol{F}}^{-T}.$$

Proof: We start with the definition of the incremental tensor, and then apply Proposition 1. After this we start substituting all our results. The derivation will be lengthy, but mostly a sequence of persistence.

$$\begin{aligned}\frac{\partial \Delta \boldsymbol{L}}{\partial \boldsymbol{F}} &= \frac{\partial}{\partial \boldsymbol{F}}\left(\Delta \boldsymbol{F} \cdot \widehat{\boldsymbol{F}}^{-1}\right) \\ &= \left(\boldsymbol{I} \looparrowright \left(\widehat{\boldsymbol{F}}^{-1}\right)^{-T}\right):\frac{\partial \Delta \boldsymbol{F}}{\partial \boldsymbol{F}} + \left(\Delta \boldsymbol{F} \looparrowright \boldsymbol{I}\right):\frac{\partial \widehat{\boldsymbol{F}}^{-1}}{\partial \boldsymbol{F}} \\ &= \left(\boldsymbol{I} \looparrowright \widehat{\boldsymbol{F}}^{-T}\right) - \frac{1}{2}\left(\Delta \boldsymbol{F} \looparrowright \boldsymbol{I}\right):\left(\widehat{\boldsymbol{F}}^{-1} \looparrowright \widehat{\boldsymbol{F}}^{-T}\right) \\ &= \left(\boldsymbol{I} \looparrowright \widehat{\boldsymbol{F}}^{-T}\right) - \frac{1}{2}\left(\Delta \boldsymbol{F} \cdot \widehat{\boldsymbol{F}}^{-1} \looparrowright \widehat{\boldsymbol{F}}^{-T}\right) \\ &= \left(\mathbf{I} - \frac{1}{2}\Delta \boldsymbol{F} \cdot \widehat{\boldsymbol{F}}^{-1}\right) \looparrowright \widehat{\boldsymbol{F}}^{-T} \\ &= \left(\widehat{\boldsymbol{F}} \cdot \widehat{\boldsymbol{F}}^{-1} - \frac{1}{2}\Delta \boldsymbol{F} \cdot \widehat{\boldsymbol{F}}^{-1}\right) \looparrowright \widehat{\boldsymbol{F}}^{-T} \\ &= \left(\left(\widehat{\boldsymbol{F}} - \frac{1}{2}\Delta \boldsymbol{F}\right) \cdot \widehat{\boldsymbol{F}}^{-1}\right) \looparrowright \widehat{\boldsymbol{F}}^{-T} \\ &= \left(\frac{1}{2}\boldsymbol{F}_0 + \frac{1}{2}\boldsymbol{F} - \frac{1}{2}\boldsymbol{F} + \frac{1}{2}\boldsymbol{F}_0\right) \cdot \widehat{\boldsymbol{F}}^{-1} \looparrowright \widehat{\boldsymbol{F}}^{-T} \\ &= \boldsymbol{F}_0 \cdot \widehat{\boldsymbol{F}}^{-1} \looparrowright \widehat{\boldsymbol{F}}^{-T}.\end{aligned}$$

Proposition 11. The map $f: V \to V$, with given invertible tensors $\boldsymbol{F}, \boldsymbol{F}_0, \boldsymbol{F} + \boldsymbol{F}_{\mathbf{0}} \in V$

$$\mathrm{f}(\mathbf{F}) = (\mathbf{F} - \mathbf{F}_0) \cdot \left(\frac{1}{2}(\mathbf{F} + \mathbf{F}_0)\right)^{-1}$$

Is invertible.

Proof: It is clear that

$$\left(\frac{1}{2}(\mathbf{F} + \mathbf{F}_0)\right)^{-1}$$

Is invertible, since its inverse is explicitly given by $\frac{1}{2}(\boldsymbol{F} + \boldsymbol{F}_0)$. Now since $f(\boldsymbol{F}) = \Delta\boldsymbol{L}$, we thus get

$$\begin{aligned}
\Delta\boldsymbol{L} = (\boldsymbol{F} - \boldsymbol{F}_0) \cdot \left(\frac{1}{2}(\mathbf{F} + \mathbf{F}_0)\right)^{-1} &\quad \Leftrightarrow \\
\frac{1}{2}\Delta\mathbf{L} \cdot (\mathbf{F} + \mathbf{F}_0) = \boldsymbol{F} - \boldsymbol{F}_0 &\quad \Leftrightarrow \\
\frac{1}{2}\Delta\boldsymbol{L} \cdot \boldsymbol{F} + \frac{1}{2}\Delta\boldsymbol{L} \cdot \boldsymbol{F}_0 + \boldsymbol{F}_0 = \boldsymbol{F} &\quad \Leftrightarrow \\
\frac{1}{2}\Delta\boldsymbol{L} \cdot \boldsymbol{F}_0 + \boldsymbol{F}_0 = \boldsymbol{F} - \frac{1}{2}\Delta\boldsymbol{L} \cdot \boldsymbol{F} &\quad \Leftrightarrow \\
\left(\frac{1}{2}\Delta\boldsymbol{L} + \boldsymbol{I}\right) \cdot \boldsymbol{F}_0 = \left(\boldsymbol{I} - \frac{1}{2}\Delta\boldsymbol{L}\right) \cdot \boldsymbol{F} &
\end{aligned}$$

Now when $\boldsymbol{I} - \frac{1}{2}\Delta\boldsymbol{L}$ is invertible, then clearly we can have $\boldsymbol{F}$ expressed as a function of $\Delta\boldsymbol{L}$, and the constant tensor $\boldsymbol{F}_0$, since then we have

$$\boldsymbol{F} = \left(\boldsymbol{I} - \frac{1}{2}\Delta\boldsymbol{L}\right)^{-1} \cdot \left(\frac{1}{2}\Delta\boldsymbol{L} + \boldsymbol{I}\right) \cdot \boldsymbol{F}_0.$$

We still need to check whether it is invertible. Let us go back to the definition of $\Delta\boldsymbol{L}$ to see that

$$\begin{aligned}
\boldsymbol{I} - \frac{1}{2}\Delta\boldsymbol{L} &= \boldsymbol{I} - \frac{1}{2}(\boldsymbol{F} - \boldsymbol{F}_0) \cdot \left(\frac{1}{2}(\boldsymbol{F} + \boldsymbol{F}_0)\right)^{-1} \\
&= \boldsymbol{I} - \frac{1}{2}(\boldsymbol{F} + \boldsymbol{F}_0 - 2\boldsymbol{F}_0) \cdot \left(\frac{1}{2}(\boldsymbol{F} + \boldsymbol{F}_0)\right)^{-1} \\
&= \boldsymbol{I} - \boldsymbol{I} + \boldsymbol{F}_0 \cdot \left(\frac{1}{2}(\boldsymbol{F} + \boldsymbol{F}_0)\right)^{-1} \\
&= \boldsymbol{F}_0 \cdot \left(\frac{1}{2}(\boldsymbol{F} + \boldsymbol{F}_0)\right)^{-1}.
\end{aligned}$$

We see that indeed $\boldsymbol{I} - \frac{1}{2}\Delta\boldsymbol{L}$ is invertible, since $\boldsymbol{F}_0$ must be invertible, and the other factor we already know to be invertible. Thus we have that the explicit inverse is given by

$$f^{-1}(\Delta\boldsymbol{L}) = \left(\boldsymbol{I} - \frac{1}{2}\Delta\boldsymbol{L}\right)^{-1} \cdot \left(\frac{1}{2}\Delta\boldsymbol{L} + \boldsymbol{I}\right) \cdot \boldsymbol{F}_0,$$

which shows that f is an invertible function.

Proposition 12. The derivative of $\boldsymbol{F}$ with respect to $\Delta\boldsymbol{L}$ is given by

$$\frac{\partial \boldsymbol{F}}{\partial \Delta\boldsymbol{L}} = \widehat{\boldsymbol{F}} \cdot \boldsymbol{F}_0^{-1} \looparrowright \widehat{\boldsymbol{F}}^T.$$

Proof: This follows directly from all the results above. We start with Proposition 11, which shows us that the map $f(\boldsymbol{F}) = \Delta\boldsymbol{L}$ is invertible, thus there exists a map f^{-1} such that $f^{-1}(\Delta\boldsymbol{L}) = \boldsymbol{F}$. Both $\boldsymbol{F}$ and $\boldsymbol{\Delta L}$ have nine independent components.

We start by considering the composition of the two maps:

$$f^{-1}\big(f(\boldsymbol{F})\big) = \boldsymbol{F},$$

thus we can take derivatives left and right, and use the chain rule to find

$$\frac{\partial f^{-1}}{\partial \Delta\boldsymbol{L}} : \frac{\partial f}{\partial \boldsymbol{F}} = \boldsymbol{I} \looparrowright \boldsymbol{I}.$$

Now we already have

$$\frac{\partial f}{\partial \boldsymbol{F}} = \boldsymbol{F}_0 \cdot \widehat{\boldsymbol{F}}^{-1} \looparrowright \widehat{\boldsymbol{F}}^{-T}.$$

So we are looking for a fourth order tensor, that when twice contracted with it gives $\boldsymbol{I} \looparrowright \boldsymbol{I}$.

Choosing

$$\frac{\partial f^{-1}}{\partial \Delta\boldsymbol{L}} = \widehat{\boldsymbol{F}} \cdot \boldsymbol{F}_0^{-1} \looparrowright \widehat{\boldsymbol{F}}^T$$

does the trick. Which we can check:

$$\begin{aligned}\left(\widehat{\boldsymbol{F}} \cdot \boldsymbol{F}_0^{-1} \looparrowright \widehat{\boldsymbol{F}}^T\right) : \left(\boldsymbol{F}_0 \cdot \widehat{\boldsymbol{F}}^{-1} \looparrowright \widehat{\boldsymbol{F}}^{-T}\right) &= \widehat{\boldsymbol{F}} \cdot \boldsymbol{F}_0^{-1} \cdot \boldsymbol{F}_0 \cdot \widehat{\boldsymbol{F}}^{-1} \looparrowright \widehat{\boldsymbol{F}}^T \cdot \widehat{\boldsymbol{F}}^{-T} \\ &= \boldsymbol{I} \looparrowright \boldsymbol{I}\end{aligned}$$

6. The Derivative of F with Respect to ΔD

This then brings us to the culmination of what we are trying to achieve, and that is to obtain the derivative of $\boldsymbol{F}$ with respect to $\Delta\boldsymbol{D}$. Since all continuum measures we are using are going to be functions of $\boldsymbol{F}$, we can then use the chain rule to see their sensitivity with respect to $\Delta\boldsymbol{D}$. The final piece of the puzzle looks rather baffling, but is in fact quite straightforward, if we consider it as a partial derivative of a full function.

To keep in line with what we discussed before, we are going to find the derivative with respect to a not necessarily symmetric tensor $\boldsymbol{D}$. But by using the symmetrisation functions, this still will work. Once limited to the space of symmetric tensors, we see that the components will then become fixed.

To finish, let $\Delta\boldsymbol{D}$ be an arbitrary symmetric tensor, and let $\Delta\boldsymbol{W}$ be an arbitrary asymmetric tensor, then $\Delta\boldsymbol{L} = \Delta\boldsymbol{D} + \Delta\boldsymbol{W}$, which is a bijective map. However, taking derivatives of a tensor with respect to a subspace tensor is difficult, so we can "undo" the projection by letting $\Delta\boldsymbol{D}$ and $\Delta\boldsymbol{W}$ be any tensors, but let $\Delta\boldsymbol{L}$ depend only on the symmetric and asymmetric parts of $\Delta\boldsymbol{D}$ and $\Delta\boldsymbol{W}$, respectively. Once we take the actual symmetric part as the representative element of the class, and use it in that sense we get the desired derivative. Thus

$$\Delta\boldsymbol{L} = \mathrm{sym}(\Delta\boldsymbol{D}) + \mathrm{asym}(\Delta\boldsymbol{W}).$$

In general, since $\Delta\boldsymbol{D}$ and $\Delta\boldsymbol{W}$ are independent of one another, we have that

$$\frac{\partial\,\mathrm{asym}(\Delta\boldsymbol{W})}{\partial\Delta\boldsymbol{D}} = \boldsymbol{0}$$

And

$$\frac{\partial\,\mathrm{sym}(\Delta\boldsymbol{D})}{\partial\Delta\boldsymbol{D}} = \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} + \boldsymbol{I} \curvearrowright \boldsymbol{I}).$$

Then also we get

$$\frac{\partial\Delta\boldsymbol{L}}{\partial\Delta\boldsymbol{D}} = \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} + \boldsymbol{I} \curvearrowright \boldsymbol{I}).$$

Finally, the derivative of $\boldsymbol{F}$ with respect to $\Delta\boldsymbol{D}$ follows from the chain rule. Thus

$$\begin{aligned}\frac{\partial\boldsymbol{F}}{\partial\Delta\boldsymbol{D}} &= \frac{\partial\boldsymbol{F}}{\partial\Delta\boldsymbol{L}} : \frac{\partial\Delta\boldsymbol{L}}{\partial\Delta\boldsymbol{D}} \\ &= \left(\widehat{\boldsymbol{F}} \cdot \boldsymbol{F}_0^{-1} \looparrowright \widehat{\boldsymbol{F}}^T\right) : \frac{1}{2}(\boldsymbol{I} \looparrowright \boldsymbol{I} + \boldsymbol{I} \curvearrowright \boldsymbol{I}) \\ &= \frac{1}{2}\left(\widehat{\boldsymbol{F}} \cdot \boldsymbol{F}_0^{-1} \looparrowright \widehat{\boldsymbol{F}}^T + \widehat{\boldsymbol{F}} \cdot \boldsymbol{F}_0^{-1} \curvearrowright \widehat{\boldsymbol{F}}^T\right)\end{aligned}$$

From the above we can see that all that happened is that we symmetrized the derivative of $\boldsymbol{F}$ with respect to $\Delta\boldsymbol{L}$. It is this derivative that needs to be used in order to get the derivative of the deformation measure that you might use.

We should note that this is not a perfect derivative for a material model that depends on $\boldsymbol{F}$ since we clearly cannot influence all the components in $\boldsymbol{F}$ by the components in $\Delta\boldsymbol{D}$.

Another way to see the same derivative appear is by noting that for any symmetric quantity $\boldsymbol{A}$, which either fully depends on $\boldsymbol{F}$ or on $\boldsymbol{D}$ we must have $\delta\boldsymbol{A} = \frac{\partial\boldsymbol{A}}{\partial\boldsymbol{F}}:\delta\boldsymbol{F}$

$$\begin{aligned}
\delta\boldsymbol{A} &= \frac{\partial\boldsymbol{A}}{\partial\boldsymbol{F}}:\delta\boldsymbol{F} \\
&= \frac{\partial\boldsymbol{A}}{\partial\Delta\boldsymbol{D}}:\delta\Delta\boldsymbol{D} \\
&= \frac{\partial\boldsymbol{A}}{\partial\Delta\boldsymbol{D}}:\frac{1}{2}(\boldsymbol{I}\looparrowright\boldsymbol{I}+\boldsymbol{I}\curvearrowright\boldsymbol{I}):\delta\Delta\boldsymbol{L} \\
&= \frac{\partial\boldsymbol{A}}{\partial\Delta\boldsymbol{D}}:\frac{1}{2}(\boldsymbol{I}\looparrowright\boldsymbol{I}+\boldsymbol{I}\curvearrowright\boldsymbol{I}):\frac{\partial\Delta\boldsymbol{L}}{\partial\boldsymbol{F}}:\delta\boldsymbol{F}
\end{aligned}$$

Now since $\delta\boldsymbol{F}$ is arbitrary this must mean that

$$\begin{aligned}
\frac{\partial\boldsymbol{A}}{\partial\boldsymbol{F}} &= \frac{\partial\boldsymbol{A}}{\partial\Delta\boldsymbol{D}}:\frac{1}{2}(\boldsymbol{I}\looparrowright\boldsymbol{I}+\boldsymbol{I}\curvearrowright\boldsymbol{I}):\left(\boldsymbol{F}_0\cdot\widehat{\boldsymbol{F}}^{-1}\looparrowright\widehat{\boldsymbol{F}}^{-T}\right) \\
&= \frac{\partial\boldsymbol{A}}{\partial\Delta\boldsymbol{D}}:\frac{1}{2}\left(\boldsymbol{F}_0\cdot\widehat{\boldsymbol{F}}^{-1}\looparrowright\widehat{\boldsymbol{F}}^{-T}+\widehat{\boldsymbol{F}}^{-T}\curvearrowright\boldsymbol{F}_0\cdot\widehat{\boldsymbol{F}}^{-1}\right)
\end{aligned}$$

We can now multiply the left and right hand side by

$$\frac{1}{2}\left(\widehat{\boldsymbol{F}}\cdot\boldsymbol{F}_0^{-1}\looparrowright\widehat{\boldsymbol{F}}^{T}+\widehat{\boldsymbol{F}}\cdot\boldsymbol{F}_0^{-1}\curvearrowright\widehat{\boldsymbol{F}}^{T}\right)$$

to find that (for fun, check it with the rules)

$$\frac{\partial\boldsymbol{A}}{\partial\boldsymbol{F}}:\frac{1}{2}\left(\widehat{\boldsymbol{F}}\cdot\boldsymbol{F}_0^{-1}\looparrowright\widehat{\boldsymbol{F}}^{T}+\widehat{\boldsymbol{F}}\cdot\boldsymbol{F}_0^{-1}\curvearrowright\widehat{\boldsymbol{F}}^{T}\right) = \frac{\partial\boldsymbol{A}}{\partial\Delta\boldsymbol{D}}:\frac{1}{2}(\boldsymbol{I}\looparrowright\boldsymbol{I}+\boldsymbol{I}\curvearrowright\boldsymbol{I})$$

Which is the same that we found before. This however depends crucially upon the fact that the variations are exactly the same, and this is not necessarily so, for example when the tensor $\boldsymbol{A}$ has certain dependencies on the rotations described in $\boldsymbol{F}$. This is however the best approximation of the derivative with respect to $\Delta\boldsymbol{D}$, and thus also the best value you can get with Abaqus.

7. Example for Neo-Hookean Material

We now show a "simple" example where we make all the relevant derivations for a Neo-Hookean material. This is to show how you would go about your own derivations. We note that a total formulation is being used.

7.1. Required Basic Derivatives

The derivative of the trace of a tensor is quite simply the identity, since:

$$
\begin{aligned}
\frac{\partial \boldsymbol{A}:\boldsymbol{I}}{\partial \boldsymbol{A}} &= \boldsymbol{A}:\frac{\partial \boldsymbol{I}}{\partial \boldsymbol{A}} + \boldsymbol{I}:\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{A}} \\
&= \boldsymbol{A}:\boldsymbol{0} + \boldsymbol{I}:(\boldsymbol{I} \looparrowright \boldsymbol{I}) \\
&= \boldsymbol{I}
\end{aligned}
$$

The derivative of the determinant we will give without proof [1], since the proof is rather involved:

$$
\frac{\partial \det \boldsymbol{A}}{\partial \boldsymbol{A}} = (\det \boldsymbol{A})\boldsymbol{A}^{-T}.
$$

7.2. The Energy Density Function

So let us now look at implementing a simple neo-Hookean user material. Here then let $\boldsymbol{F}$ be the deformation gradient, and $J = \det \boldsymbol{F}$. The isochoric deformation gradient is given by $\bar{\boldsymbol{F}} = J^{-1/3}\boldsymbol{F}$. The isochoric right Cauchy Green stretch tensor is then given by $\bar{\boldsymbol{C}} = \bar{\boldsymbol{F}}^{T} \cdot \bar{\boldsymbol{F}}$. Finally $\bar{I}_1 = \bar{\boldsymbol{C}}:\boldsymbol{I}$. With these definitions in hand, an energy density function is defined, which for a neo-Hookean material takes the form:

$$
U = C_{10}(\bar{I}_1 - 3) + D_1(J-1)^2.
$$

7.3. The Stress

The stress is obtained by taking the appropriate derivative of the aforementioned energy density function. We need a derivative for J, but that is just the derivative of the determinant of $\boldsymbol{F}$. We also need the derivative of $\bar{I}_1$; this is going to be

$$
\begin{aligned}
\frac{\partial \bar{I}_1}{\partial \boldsymbol{F}} &= \frac{\partial}{\partial \boldsymbol{F}}\left(J^{-\frac{2}{3}} I_1\right) \\
&= J^{-\frac{2}{3}}\frac{\partial I_1}{\partial \boldsymbol{F}} - \frac{2}{3}J^{-\frac{5}{3}} I_1 \frac{\partial J}{\partial \boldsymbol{F}} \\
&= J^{-\frac{2}{3}}\boldsymbol{I}:\frac{\partial \boldsymbol{C}}{\partial \boldsymbol{F}} - \frac{2}{3}J^{-\frac{5}{3}} I_1 J \boldsymbol{F}^{-T} \\
&= J^{-\frac{2}{3}}\boldsymbol{I}:\left((\boldsymbol{I} \looparrowright \boldsymbol{F}^T):\frac{\partial \boldsymbol{F}^T}{\partial \boldsymbol{F}} + (\boldsymbol{F}^T \looparrowright \boldsymbol{I}):\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{F}}\right) - \frac{2}{3}\bar{I}_1 \boldsymbol{F}^{-T} \\
&= J^{-\frac{2}{3}}\boldsymbol{I}:\big((\boldsymbol{I} \looparrowright \boldsymbol{F}^T):(\boldsymbol{I} \curvearrowright \boldsymbol{I}) + (\boldsymbol{F}^T \looparrowright \boldsymbol{I}):(\boldsymbol{I} \looparrowright \boldsymbol{I})\big) - \frac{2}{3}\bar{I}_1 \boldsymbol{F}^{-T} \\
&= J^{-\frac{2}{3}}\big(\boldsymbol{I}:(\boldsymbol{I} \curvearrowright \boldsymbol{F}^T) + \boldsymbol{I}:(\boldsymbol{F}^T \looparrowright \boldsymbol{I})\big) - \frac{2}{3}\bar{I}_1 \boldsymbol{F}^{-T} \\
&= 2J^{-\frac{2}{3}}\boldsymbol{F} - \frac{2}{3}\bar{I}_1 \boldsymbol{F}^{-T}
\end{aligned}
$$

From line 1 to line 2, we used the product rule, the definition of $\bar{I}_1$, and a subsequent application Proposition 8. We then used the derivatives of transpose and self, after which we need to apply the products using the rules set out in Proposition 1. Which leaves us with contractions of second order tensors and fourth order tensors, so we follow Proposition 2 which gives us the final result. Substituting all this in our derivative of the energy density function gives us:

$$\boldsymbol{P} = \frac{\partial U}{\partial \boldsymbol{F}} = 2C_{10}\left(J^{-\frac{2}{3}}\boldsymbol{F} - \frac{1}{3}\bar{I}_1\boldsymbol{F}^{-T}\right) + 2D_1(J-1)J\boldsymbol{F}^{-T}$$

This is the first Piola-Kirchhoff stress. The Cauchy stress, as requested by Abaqus, is obtained by applying the standard push-forward operation, and thus gives us:

$$\begin{aligned}
\boldsymbol{\sigma} &= \frac{1}{J}\boldsymbol{F}\cdot\boldsymbol{P}^T \\
&= \frac{2C_{10}}{J}\left(J^{-\frac{2}{3}}\boldsymbol{F}\cdot\boldsymbol{F}^T - \frac{1}{3}\bar{I}_1\boldsymbol{I}\right) + 2D_1(J-1)\boldsymbol{I} \\
&= \frac{2C_{10}}{J}\left(\overline{\boldsymbol{B}} - \frac{1}{3}(\overline{\boldsymbol{B}}:\boldsymbol{I})\boldsymbol{I}\right) + 2D_1(J-1)\boldsymbol{I} \\
&= \frac{2C_{10}}{J}\mathrm{dev}(\overline{\boldsymbol{B}}) + 2D_1(J-1)\boldsymbol{I}
\end{aligned}$$

7.4. The Consistent Tangent

As stated in the Abaqus User Subroutines Guide, the following derivative must be defined in subroutine UMAT:

$$\begin{aligned}
\boldsymbol{\mathcal{C}} &= \frac{1}{J}\frac{\partial\Delta(J\boldsymbol{\sigma})}{\partial\Delta\boldsymbol{\varepsilon}} \\
&= \frac{1}{J}\frac{\partial(J_0\boldsymbol{\sigma}_0 + \Delta(J\boldsymbol{\sigma}))}{\partial\Delta\boldsymbol{\varepsilon}} \\
&= \frac{1}{J}\frac{\partial(J\boldsymbol{\sigma})}{\partial\boldsymbol{F}}\frac{\partial\boldsymbol{F}}{\partial\Delta\mathbf{D}}
\end{aligned}$$

Where we go from the first to the second line by observing that $\boldsymbol{\sigma}_0$ and $\Delta\boldsymbol{\sigma}$ are in the same configuration, and the initial stress multiplied by the initial volume stretch is a constant within the increment; thus its derivative with respect to $\Delta\boldsymbol{\varepsilon}$ is equal to zero, so we are free to add it in the numerator. (We are free to add any constant).

We now need the derivative in the middle of the last expression. This is written out fully as:

$$\begin{aligned}
\frac{\partial(J\boldsymbol{\sigma})}{\partial\boldsymbol{F}} &= \frac{\partial}{\partial\boldsymbol{F}}\left(2C_{10}\mathrm{dev}(\overline{\boldsymbol{B}}) + 2D_1(J^2-J)\boldsymbol{I}\right) \\
&= 2C_{10}\frac{\partial\mathrm{dev}(\overline{\boldsymbol{B}})}{\partial\boldsymbol{F}} + 2D_1(2J-1)\boldsymbol{I}\otimes J\boldsymbol{F}^{-T}
\end{aligned}$$

We are now left with computing

$$
\begin{aligned}
\frac{\partial \mathrm{dev}(\overline{\boldsymbol{B}})}{\partial \boldsymbol{F}} &= \left(\boldsymbol{\mathcal{J}} - \frac{1}{3}\boldsymbol{I}\otimes\boldsymbol{I}\right):\frac{\partial \overline{\boldsymbol{B}}}{\partial \boldsymbol{F}} \\
&= \left(\boldsymbol{\mathcal{J}} - \frac{1}{3}\boldsymbol{I}\otimes\boldsymbol{I}\right):\left(J^{-\frac{2}{3}}\frac{\partial \boldsymbol{F}\cdot\boldsymbol{F}^{T}}{\partial \boldsymbol{F}} - \frac{2}{3}J^{-\frac{5}{3}}\boldsymbol{B}\otimes J\boldsymbol{F}^{-T}\right) \\
&= \left(\boldsymbol{\mathcal{J}} - \frac{1}{3}\boldsymbol{I}\otimes\boldsymbol{I}\right):\left(J^{-\frac{2}{3}}\left((\boldsymbol{I}\looparrowright\boldsymbol{F}):\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{F}} + (\boldsymbol{F}\looparrowright\boldsymbol{I}):\frac{\partial \boldsymbol{F}^{T}}{\partial \boldsymbol{F}}\right) - \frac{2}{3}\overline{\boldsymbol{B}}\otimes\boldsymbol{F}^{-T}\right) \\
&= \left(\boldsymbol{\mathcal{J}} - \frac{1}{3}\boldsymbol{I}\otimes\boldsymbol{I}\right):\left(J^{-\frac{2}{3}}\left((\boldsymbol{I}\looparrowright\boldsymbol{F}) + (\boldsymbol{F}\curvearrowright\boldsymbol{I})\right)\right) - \frac{2}{3}\mathrm{dev}(\overline{\boldsymbol{B}})\otimes\boldsymbol{F}^{-T} \\
&= J^{-\frac{1}{3}}\left(\boldsymbol{I}\looparrowright\overline{\boldsymbol{F}} + \overline{\boldsymbol{F}}\curvearrowright\boldsymbol{I} - \frac{2}{3}\boldsymbol{I}\otimes\overline{\boldsymbol{F}}\right) - \frac{2}{3}\mathrm{dev}(\overline{\boldsymbol{B}})\otimes\boldsymbol{F}^{-T}
\end{aligned}
$$

Now to multiply this by $\frac{1}{2}\left(\widehat{\boldsymbol{F}}\cdot\boldsymbol{F}_0^{-1}\looparrowright\widehat{\boldsymbol{F}}^{T} + \widehat{\boldsymbol{F}}\cdot\boldsymbol{F}_0^{-1}\curvearrowright\widehat{\boldsymbol{F}}^{T}\right)$ we get the derivative with respect to $\Delta\boldsymbol{\varepsilon}$. We will not further simplify this evaluation; we implement all directly in the example code.

7.5. Result

The example code is outlined below and included in the example files. The tensor ordering for representing any tensor as a vector is given as $F_{xx}, F_{yy}, F_{zz}, F_{xy}, F_{xz}, F_{yz}, F_{yx}, F_{zx}, F_{zy}$. This is the reordering that is implicitly being performed in the weave, clamp and outer product routines. This allows us to do extraction of portions of the fourth order stiffness tensor, and also how we extract the vectorial stress from the matrix represented stress.

```fortran
subroutine umat(stress,sdv,ddsdde,sse,spd,scd,rpl,ddsddt,                      &
                drplde,drpldt,stran,dstran,time,dtime,temp,dtemp,predef,dpred, &
                cmname,ndi,nshr,ntens,numsdv,props,nprop, coords,drot,pnewdt,  &
                celent,F0,F,noel,npt,layer,kspt,kstep,kinc)
    ! ..
    use kPrecision   , only : rp
    use kConstants
    use kLisa
    use kTensorManipulation
    ! ..
    implicit none
    ! ..
    ! Parameters
    integer, parameter :: iStepTime       = 1,    &
                          iTotalTime      = 2,    &
                          nTime           = 2
    ! ..
    ! Arguments
    ! ..
    character*80,intent(in)      :: cmname
    real(rp), intent(inout)      :: stress(ntens), sdv(numsdv)
    real(rp), intent(inout)      :: ddsddt(ntens), ddsdde(ntens, ntens)
    real(rp), intent(inout)      :: drplde(ntens), drpldt
    real(rp), intent(inout)      :: sse, spd, scd, rpl, pnewdt
    real(rp), intent(in)         :: stran(ntens), dstran(ntens)
    real(rp), intent(in)         :: time(2)     , dtime
    real(rp), intent(in)         :: temp        , dtemp
    real(rp), intent(in)         :: predef(2)   , dpred(2)
    real(rp), target, intent(in) :: props(nprop)
    real(rp), intent(in)         :: coords(3)   , drot(3,3)
    real(rp), intent(in)         :: F0(3,3)     , F(3,3), celent
    integer,  intent(in)         :: ndi, nshr, ntens, numsdv, nprop
    integer,  intent(in)         :: noel,npt,layer,kspt,kstep,kinc
    ! ..
    ! Locals
    ! ..
    real(rp) :: I1bar, Fbar(3,3), Bbar(3,3), devBbar(3,3), J, J1_3, I(3,3)
    real(rp) :: sig(3,3), dJsigdF(9,9), dJsigddD(9,9), Fhat(3,3), F0inv(3,3), FhatF0inv(3,3)
    real(rp) :: C10, D1
    ! ..
    ! Code
    ! ..
    C10 = props(1)
    D1  = props(2)
    ! ..
    J     = kDet(F)
    J1_3  = J**third
    Fbar  = F/J1_3
    Bbar  = matmul(Fbar,transpose(Fbar))
    I1bar = Bbar(1,1) + Bbar(2,2) + Bbar(3,3)
    I     = kEye(3)
    ! ..
    ! Time to compute U
    sse = C10*(I1bar - three) + D1*(J - one)*(J - one)
    ! ..
    ! And the derivative of U
    devBbar = Bbar - I1bar/three*I
    sig = two*C10/J*devBbar + two*D1*(J-one)*I
    ! ..
    ! And then the stiffness
    dJsigdF = two*C10/J1_3*(kWeave(I, Fbar) + kClamp(Fbar, I) - two/three*kOuter(I, Fbar))
    dJsigdF = dJsigdF + kOuter(two*D1*(two*J-one)*J*I-four/three*C10*devBbar, transpose(kInvert(F)))
    ! ..
    ! Stiffness multiplier
    Fhat  = half*(F+F0)
    F0inv = kInvert(F0)
    FhatF0inv = matmul(Fhat, F0inv)
    dJsigddD = matmul(dJsigdF, half*(kWeave(FhatF0inv, transpose(Fhat)) + kClamp(FhatF0inv, transpose(Fhat))))
    ! ..
    ! Onto the stress and stiffness
    stress = kVector(sig, ntens)
    ddsdde = kSym4(dJsigddD, ntens, .true.)/J
end subroutine umat
```

Figure 1. Neo Hookean code

8. Conclusion

We have shown how to obtain the derivative when we wish to use an alternative strain measure than the one Abaqus is using. This is given by a transformation matrix as

$$\frac{\partial \boldsymbol{F}}{\partial \Delta \boldsymbol{D}} = \frac{1}{2}\left(\widehat{\boldsymbol{F}} \cdot \boldsymbol{F}_0^{-1} \looparrowright \widehat{\boldsymbol{F}}^T + \widehat{\boldsymbol{F}} \cdot \boldsymbol{F}_0^{-1} \sim \widehat{\boldsymbol{F}}^T \right).$$

In order to understand the above equation, we introduced some fourth order tensor construction operators, as well as the rules that facilitate their use. We hope that, even though this white paper is quite theoretical, this will allow the material programming enthusiast to derive the correct derivatives.

To show how a subroutine will look with just this line of code inserted, we've added an example of a neo-Hookean material, including how the derivative of this material can be obtained using all the tools provided in the paper.

9. References

1. Magnus, Jan R.; Neudecker, Heinz (1999). Matrix Differential Calculus with Applications in Statistics and Econometrics. Wiley. ISBN 0-471-98633-X

Abaqus documentation references:

2. "User-defined mechanical material behavior," Abaqus 2017 Materials Guide
3. "UMAT," Abaqus 2017 User Subroutines Guide

For more general information on Abaqus user defined materials:

4. Dunne, F., & Petrinic, N., 2005, Introduction to Computational Plasticity, Oxford UK, Oxford University Press
5. Weber, G., & Anand, L., "Finite Deformation Constitutive Equations and a Time Integration Procedure for Isotropic, Hyperelastic-Viscoplastic Solids," Computer Methods in Applied Mechanics and Engineering, v. 79, pp. 173-202, 1990.
6. Aravas, N., "On the Numerical Integration of a Class of Pressure-Dependent Plasticity Models," International Journal for Numerical Methods in Engineering, v. 24, pp. 1395-1416, 1987.

10. Document History

Document Revision	Date	Revised By	Changes/Notes
1.0	19 December 2017	David PALMER	First Release